Multi-Item Inventory Control: The K-Curve Methodology

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Abstract: This article deals with the problem of optimally coordinating the replenishment in a multi-item stock-keeping environment. The K-Curve Methodology represents a way to classify all items into a given number of classes with a common order frequency. A theoretical investigation of the K-Curve Methodology however shows that this methodology does not achieve an optimal grouping of the items in respect to the cost function considered in most articles. Practical experience shows that the K-Curve Methodology fits the objectives of management and represents a good decision aid in practice.

1 Introduction

A multi-item inventory system is considered which has the property that, for each single item, a reorder policy using the EOQ formula would be appropriate. The ordering costs can not be reduced by ordering items jointly.

In real-life procurement processes it is often desirable to form a number of classes of items where all items of one class share the same order cycle. This yields simplified stock control, better supplier relations, more reliable deliveries and improved warehouse planning [Agg83]. Practitioners who are responsible for the management of thousands of items have to define objectives regarding inventory level, workload, and customer service and furthermore simple methodologies are needed to take decisions.

The question which items should have the same order cycle in order to minimize the total costs associated with ordering and holding the items given a desired number of groups has been studied by several authers. Page and Paul [PP76] were the first to consider a coordinated replenishment of items and proposed an algorithm for classifying the items. Chakravarty showed that the optimal groups possess a consecutiveness property and hence can be found by using a dynamic programming recursion [Cha81] or a shortest-path algorithm [COR82]. Donaldson [Don81] suggested to choose the ordering frequencies in geometric progression. Aggarwal [Agg83] assumed that the cumulative distribution by values of the inventory can be characterized by a Pareto function of a certain type and established optimal boundaries of the groups in closed-form. Aggarwal studied in [Agg84c] the sensitivity of the optimal group boundaries so obtained. The results demonstrate the availability of flexibility in the partition boundaries. In [Agg84a] Aggarwal proposed a computationally efficient procedure that determines the optimal order cycle values when no information about the value distribution is at hand. Chakravarty [Cha85] developed a fast heuristic if setup costs are taken into account. This heuristic was based on a theorem stating a consecutiveness property of the classes. Bastian [Bas86] proved Chakravarty's theorem and showed how the necessary optimality criteria that have been proposed by Chakravarty have to be modified. Crouch and Oglesby [CO78] considered the problem of grouping the items with a common lot size for each group, when the value distribution is lognormal. Rules for obtaining the optimal ratios between lot sizes are de-

fined. Their results can be applied to the case of single replenishment frequencies for all items of one group. Extended reviews about an indirect grouping strategy where the replenishment cycle of each item (or group) is an integer multiple of a basic cycle time can be found in [AE88] and [GS89]. This paper argues that practitioners should establish another inventory management system. It is called the K-Curve Methodology (KCM) and has been developed by Shah, Bucher and Relph [SBR90]. The KCM represents a decision aid that helps management to achieve a balance between inventory and workload. The KCM is basically a periodic order cycle inventory management system. The items are classified into a given number of classes with a common order frequency for all items. By flexing the cost ratio it is possible to predict the total inventory performance.
The paper is organised as follows: in section 2 the optimisation problem that has been considered in the literature is described. Section 3 introduces the K-Curve Methodology. Theoretical investigations of the optimisation problem are presented in section 4. A comparison of the advantages of the theoretically optimal solution and the KCM in section 5 reveal that a controlled application of the KCM mostly fits the objectives of management.

2 The Optimisation Problem

2.1 Assumptions and Notation

Consider a multi-item inventory system with n independent items $P_1, ..., P_n$. For each item it is assumed that the deterministic E.O.Q. model can be used. In other words all items can be ordered in non negative quantities of any size. The demand for individual items is known to occur deterministic with stationary rate. A_i units of item $P_i, i = 1, ..., n$ are demanded per unit of time over an infinite time horizon. Shortages are not allowed. Orders can be placed at any time. The order lead-time is zero (or a positive constant). The yield on an order is 100%. All costs are stationary and consist of a fixed ordering cost C that is assumed to be the same for all items and a proportional ordering cost V_i per unit of item $P_i, i = 1, ..., n$. A proportional holding cost I per unit of time is charged for each dollar tied up in stock. Constraints on any resources that are used do not exist. In this model ordering items jointly does not reduce ordering costs.
Letting $F_i \in R_+$ denote the frequency of replenishments per unit of time for item $P_i, i = 1, ..., n$, the objective function $\mathrm{TC_i}$ of product P_i is the sum of the expected replenishment and holding costs per unit of time (see [Ter94] p. 92)

$$\mathrm{TC_i}(F_i) = CF_i + I\frac{A_i V_i}{2F_i}.$$

$\mathrm{TC_i}$ is strictly convex in F_i and the global minimum is located in $F_i^* = \sqrt{\frac{IA_iV_i}{2C}}$. The product A_iV_i represents the annual usage value of item P_i and will be denoted by $\mathrm{AUV_i} = A_i V_i, i = 1, ..., n$. Replacing the costs by a cost ratio

$$K = \frac{2C}{I}, \tag{1}$$

F_i^* can be expressed alternatively as

$$F_i^* = \sqrt{\frac{\mathrm{AUV_i}}{K}}. \tag{2}$$

The economic order quantity is given by $Q_i^* = \frac{A_i}{F_i^*} = \sqrt{\frac{KA_i}{V_i}}$ and the optimal value of the objective function is $\mathrm{TC_i^*} = \sqrt{2CI\mathrm{AUV_i}}$.

A partition of the set $S = \{P_1, ..., P_n\}$ into m subsets will be denoted by $\mathcal{S} = (S_1, ..., S_m)$. The set of all partitions of S is denoted by $\mathcal{P}$, the number of items in set S_j by $|S_j|$. The order frequencies of the m classes are given by $\mathrm{F} := (\mathrm{F}_1, ..., \mathrm{F}_\mathrm{m})$.

2.2 Optimisation Model

We consider the problem of classifying the inventory items $S = \{P_1, ..., P_n\}$ into a given number of classes $\mathcal{S} = (S_1, ..., S_m)$ with a common order frequency for each class, so as to minimize the inventory (holding and ordering) costs (see [Cha81], [Agg83], [Bas86]). The total cost function TC for the inventory system consists of the sum of the total expected replenishment and holding costs for each class of items per unit of time. TC is a function of the partition $\mathcal{S} \in \mathcal{P}$ and the order frequencies $\mathrm{F} := (\mathrm{F}_1, ..., \mathrm{F}_\mathrm{m}) \in \mathrm{R}_+^\mathrm{m}$:

$$\mathrm{TC}(\mathcal{S}, \mathrm{F}) = C \sum_{j=1}^{m} |S_j| F_j + I \sum_{j=1}^{m} \frac{\sum_{i \in S_j} \mathrm{AUV_i}}{2F_j}.$$

Define $\mathrm{AUV}_{S_j} := \sum_{i \in S_j} \mathrm{AUV_i}$. Multiplication of TC with $\frac{2}{I}$ and introduction of $K = \frac{2C}{I}$ from (1) leads to the objective function

$$\mathrm{Z}(\mathcal{S}, \mathrm{F}) = K \sum_{j=1}^{m} |S_j| F_j + \sum_{j=1}^{m} \frac{\mathrm{AUV}_{S_j}}{F_j} = \sum_{j=1}^{m} \mathrm{Z_j}(\mathrm{S_j}, \mathrm{F_j}) \tag{3}$$

where

$$\mathrm{Z_j}(S_j, F_j) = K |S_j| F_j + \frac{\mathrm{AUV}_{S_j}}{F_j}. \tag{4}$$

2.3 Optimality Conditions

Given any partition $\mathcal{S} = (S_1, ..., S_m)$ of S the optimal order frequency F_j for the items in class S_j can be determined. Only the summand $\mathrm{Z_j}$ of Z depends on S_j or F_j (see (3) and (4)).

$$\mathrm{Z_j}(S_j, F_j) = K |S_j| F_j + \mathrm{AUV}_{S_j} \frac{1}{F_j}$$

is strictly convex in F_j. The optimal order frequency of class S_j is therefore given by

$$F_j^* = \sqrt{\frac{\mathrm{AUV}_{S_j}}{K |S_j|}}. \tag{5}$$

Upon substitution for $F_1, ..., F_m$ in (3) with the optimal values from (5) the optimisation problem of partitioning n items into a prescribed number of m sets is given by

$$\mathrm{Z}(\mathcal{S}) \quad = \quad \sum_{j=1}^{m} \sqrt{K |S_j| \mathrm{AUV}_{S_j}} \rightarrow \min_{\mathcal{S} \in \mathcal{P}}. \tag{6}$$

This shows that the optimal partition for the items is independent of the cost ratio K. The optimal partition only depends on the annual usage values of the items. The number of partitions that have to be considered is

$$|\mathcal{P}| = \sum_{k=1}^{m} S(n, k),$$

where $S(n,k) := \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$ are the *Stirling numbers of the second kind.*
By a theorem of Chakravarty [Cha81] this number can be reduced.

Definition 2.1 *A set S_j is called **consecutive** if and only if $u, v \in S_j, w \in S, \mathrm{AUV_u} < \mathrm{AUV_w} < \mathrm{AUV_v}$ implies $w \in S_j$.*

Theorem 2.2 *There exists an optimal grouping of items such that all sets S_j are consecutive.*

From (5) we conclude

Corollary 2.3 *The optimal order frequencies of two consecutive sets S_j and S_{j+1} with $\min\{\mathrm{AUV_i} | \mathrm{i} \in \mathrm{S_j}\} > \max\{\mathrm{AUV_i} | \mathrm{i} \in \mathrm{S_{j+1}}\}$ have the property $F_j^* > F_{j+1}^*$.*

Knowing that we may confine our search for optimal solutions to consecutive sets we assume that the items are numbered in decreasing order of their annual usage values. If the annual usage values of the items are all different, the problem is to determine $m-1$ values $r_1, ..., r_{m-1}$ (boundaries) expressed in the dimension of an annual usage value which are separating the classes. The number of partitions that have to be considered is lowered significantly and is given by $\binom{n-1}{m-1}$.

3 The K-Curve Methodology

Shah, Bucher and Relph [SBR90] have introduced the K-Curve Methodology (KCM). It is based on the economic order quantity (EOQ) and the underlying cost structure (see [Ter94]). The EOQ has been dismissed by many practitioners because of its reliance on ordering and inventory holding costs C and I. The values of these costs are difficult if not impossible to determine [LR95]. The KCM therefore replaces C and I by the controlling parameter $K = \frac{2C}{I}$, which can be seen as a decision variable for management.

Given a list of items $S = \{P_1, ..., P_n\}$ and an annual usage value $\mathrm{AUV_i}$ for each item $P_i, i = 1, ..., n$, the KCM establishes a grouping of the items as follows: the number of classes m and the order frequencies $F_1, ..., F_m$ of the classes $S_1, ..., S_m$ are prescribed in form of a decreasing series. Having sorted the items by their annual usage values the class boundary between class S_j and S_{j+1} is set as KF_jF_{j+1}. Now, all items with usage values between KF_1F_2 and infinity belong to S_1, items with usage values between KF_1F_2 and KF_2F_3 belong to S_2 and so on. At this stage the actual number of orders and the average inventory expected which are the important variables of inventory management are calculated according

$$\begin{aligned} \text{number of orders} &= \sum_{j=1}^{m} |S_j| \cdot F_j, \\ \text{average inventory} &= \sum_{j=1}^{m} \frac{\sum_{P_i \in S_j} \mathrm{AUV_i}}{2F_j}, \end{aligned}$$

where the inner sum is taken over all items P_i, $i = 1, ..., n$ that belong to class S_j.

By substituting different values of K into the formula other groupings arise. Each value of K generates a different number of orders and average inventory. The results can be plotted as a graph. This Exchange Curve (K-Curve) shows the balance between the adverse requirements of low inventory and a small number of orders that can be achieved by using the KCM. Using the K-Curve management can decide the required position on the curve and the parameters corresponding to this position can be implemented.

In a second paper Relph et al. [RBD94] found by using heuristic arguments that choosing the order frequencies as a geometric progression with factor 2 performs well. The optimal number of classes in their setting was eight. The inventory performance of the optimum series of order frequencies was independent of the industry type, the number of parts and the average value per part. A recent paper about the KCM gives an extension of this methodology to the case where setup costs are involved [BDR95].

3.1 First Investigation of the KCM

At this stage the question arises if the KCM produces an optimal grouping of the items in respect of the optimisation problem developed in section 2. We can deny the optimality of the grouping produced by the KCM with respect to the cost structure of the EOQ (which was used by Chakravarty [Cha81], Aggarwal [Agg83] and Bastian [Bas86]) at once. This is due to the fact that the KCM classifies all items according to the value of K that has been chosen appropriate to represent the inventory performance. The objective function Z in (6) showed that the optimal grouping is independent of the cost ratio K. Therefore the optimal grouping only depends on the distribution of the annual usage values.
Theorem 2.2 shows that the KCM uses the right approach when sorting the items by their annual usage values. The boundaries between the classes are therefore worth further investigations.

Theorem 3.1 (Optimal class boundaries) *Given $F_1 > ... > F_m$ the optimal boundary r_j between S_j and S_{j+1} expressed in the dimension of an annual usage value is given by $r_j = KF_jF_{j+1}$ for all $j = 1, ..., m-1$.*

Proof. The idea for the proof is adapted from Donaldson [Don81]. Consider item P_i which has an annual usage value of $\mathrm{AUV_i}$. In the EOQ model the objective function of this item is given by $\mathrm{Z}(F) = KF + \frac{\mathrm{AUV_i}}{F}, F \in R$. This function is minimised by $F^* = \sqrt{\frac{\mathrm{AUV_i}}{K}}$ with $\mathrm{Z}(F^*) = 2KF^*$. Choosing any other order frequency F leads to $\mathrm{Z}(F) = K(F + \frac{F^{*2}}{F}) > \mathrm{Z}(F^*)$.
Assume that for S_j and S_{j+1} the order frequencies F_j and F_{j+1} with $F_j > F_{j+1}$ are given and that the optimal order frequency of P_i is F^* with $F_j > F^* > F_{j+1}$. It is of no importance into which of the two classes P_i falls if one of the following equivalent conditions holds

$$\begin{aligned}
\mathrm{Z}(F_j) &= \mathrm{Z}(F_{j+1}) \\
K(F_j + \frac{F^{*2}}{F_j}) &= K(F_{j+1} + \frac{F^{*2}}{F_{j+1}}) \\
F^{*2}(\frac{1}{F_j} - \frac{1}{F_{j+1}}) &= F_{j+1} - F_j \\
F^* &= \sqrt{F_jF_{j+1}}.
\end{aligned}$$

The optimal lower boundary of class S_j expressed in the dimension of an optimal order frequency is $\sqrt{F_jF_{j+1}}$. Expressed in the dimension of an annual usage value the lower boundary of S_j is given by $r_j = KF_jF_{j+1}$. □

Theorem 3.1 shows that all items with usage values between r_j and r_{j-1} should belong to class S_j. This means that the KCM prescribes the optimal class boundaries. The order frequencies are certainly not object of an optimisation.
Letting $\bar{r}_j := \frac{\mathrm{AUV}_{S_j}}{|S_j|}$ denote the mean annual usage value of the items in class S_j we obtain the following (see (5))

Corollary 3.2 *The optimal boundary r_j between S_j and S_{j+1} with order frequencies F_j^* and F_{j+1}^* is given by $r_j = \sqrt{\bar{r}_j\bar{r}_{j+1}}$.*

3.2 Reformulation of the Optimization Problem

Given $F_1, ..., F_m$, the optimal class boundaries are $r_j = KF_jF_{j+1}$ for all $j = 1, ..., m-1$. If, on the other hand, the boundaries $r_1, ..., r_{m-1}$ are given, the optimal order frequencies $F_1, ..., F_m$ can be ascertained using (5). The optimisation problem for prescibed K is to determine either the optimal order frequencies $F_1 > F_2 > ... > F_m$ or the optimal class boundaries $r_1 > ... > r_{m-1}$, where in both cases the other decision parameter is implicitly determined optimally.

The optimisation problem in the first case is given by

$$\begin{aligned} \mathrm{Z_F(F)} &= \sum_{j=1}^{m}\left(K|S_j|F_j + \frac{\mathrm{AUV}_{S_j}}{F_j}\right) && (7)\\ &= \sum_{j=1}^{m}\sum_{i=1}^{n}\left(KF_j 1_{\{\mathrm{AUV_i}\in[r_j, r_{j-1})\}} + \frac{\mathrm{AUV_i} 1_{\{\mathrm{AUV_i}\in[r_j, r_{j-1})\}}}{F_j}\right) \rightarrow \min_{\mathrm{F}\in \mathrm{R}^{\mathrm{m}}_{+,\geq}} \end{aligned}$$

with $r_j = KF_jF_{j+1}$, $j = 1, ..., m-1$, $r_0 = \infty$, $r_m = 0$ where $R^m_{+,\geq} := \{x \in R^m_+ : x_1 \geq ... \geq x_m\}$ and 1_X represents the characteristic function of the set X.

In the second case we obtain

$$\begin{aligned} \mathrm{Z_r}(r_1, ..., r_{m-1}) &= \sum_{j=1}^{m}\sqrt{K|S_j|\mathrm{AUV}_{S_j}} && (8)\\ &= \sum_{j=1}^{m}\sqrt{K\sum_{i=1}^{n} 1_{\{\mathrm{AUV_i}\in[r_j, r_{j-1})\}} \sum_{i=1}^{n}\mathrm{AUV_i} 1_{\{\mathrm{AUV_i}\in[r_j, r_{j-1})\}}} \rightarrow \min_{(r_1, ..., r_{m-1})\in R^{m-1}_{+,\geq}} . \end{aligned}$$

Several heuristic as well as optimal algorithms for the creation of groups have been proposed. Chakravarty used a dynamic programming recursion [Cha81] and a shortest-path algorithm [COR82]. Both algorithms make use of the consecutiveness property of the classes and determine an optimal classification of the items. Their run time is given by $O(mn^2)$. Heuristic algorithms which need less time to find a good solution have been proposed by Page/Paul [PP76], Aggarwal [Agg84a], Chakravarty [Cha85] and Bastian [Bas86]. They have several disadvantages with regard to computational effort and optimality of the grouping. As far as our experience goes the following algorithm is very promising

Initialization:
Let S^0 be a starting partition of S.
Iteration:

1. The partition $S^v, v > 0$ is constructed as follows: for $j = 1, ..., m$ find $F_{j_{new}}$ (for example with binary search), such that the new partition with new boundaries $r_j = KF_{j_{new}}F_{j+1}, r_{j-1} = KF_{j+1}F_{j_{new}}$ of class S_j achieves the best improvement with respect to the objective function Z.

2. Stop if $|F_{j_{new}}F_j| < \epsilon_1$ for $j = 1, ..., m$ or $\mathrm{Z}(S^v) - \mathrm{Z}(S^{v+1}) < \epsilon_2$.

4 Further investigation of the optimisation problem for theoretical distributions of the annual usage values

4.1 Description of the model

The optimal classification into a given number of groups only depends on the annual usage values of the items in consideration but not on the costs involved. The algorithms that produce classifications of the items are computationally unattractive with large-size problems. Further insight into the problem under consideration can be achieved by assuming that the distribution of the annual usage value of the items can be characterised by a continuous distribution function. If the real distribution by value curve (DBV Curve) of the inventory is approximated by a continuous distribution function some heuristics can be proposed that need less computational time.

We introduce the continuous version of the optimisation problem. Instead of the discrete and finite set $(\mathrm{AUV}_1, ..., \mathrm{AUV}_\mathrm{n})$ a real interval $T = [r_m, r_0] \subset \overline{R}_+$ has to be partitioned optimally into m classes. The formal description for the distribution by value curve follows Aggarwal [Agg84b]. Assume that the annual usage value $r = \mathrm{AUV} \in T$ is random and that it can be represented by a random variable $R : (\Omega, \mathcal{A}, P) \to (T, B \cap T)$ with finite mean μ_R. Let F be the distribution function of R, i.e. $F(r) = P\{R \leq r\}$. F being differentiable on T, and define $f(r) := \frac{dF(r)}{dr} \geq 0$, $r \in T$; so $F(r) = \int_{r_m}^{r} f(x)\,dx$. Subsequently we assume that the density f is strictly positive on T. The theoretical analogue of the empirical DBV Curve is defined as follows: for all $r \in T$

$$
\begin{aligned}
X(r) &:= \int_r^{r_0} f(x)\,dx && \text{fraction of all items that have values} \geq r, \\
Y(r) &:= \frac{1}{\mu_R}\int_r^{r_0} x f(x)\,dx && \text{fraction of total value contributed by items with value} \geq r.
\end{aligned}
$$

Y and X are strictly monotonically decreasing in r with $0 \leq X(r), Y(r) \leq 1$.

Definition 4.1 *The* ***DBV Curve*** Γ *is a curve in* $[0,1]^2$ *given by* $\Gamma = \{\gamma(r) : r \in [r_m, r_0]\}$ *where*

$$
\begin{aligned}
\gamma :\ [r_m, r_0] &\to [0,1]^2 \\
r &\to (X(r), Y(r)).
\end{aligned}
$$

Every value of $r \in T$ determines exactly one point on the DBV Curve Γ. X is continuous and strictly monotonically decreasing with $X(r_m) = 1$ and $X(r_0) = 0$. It follows that X is bijective and its inverse $X^{-1} : [0,1] \to [r_m, r_0]$ is continuous and strictly monotonically decreasing as well. The composition

$$
\begin{aligned}
\gamma' := \gamma \circ X^{-1} :\ [0,1] &\to [0,1]^2 \\
n &\to (n, Y(X^{-1}(n)))
\end{aligned}
$$

represents a path in $[0,1]^2$ which results from γ by parameter transformation X^{-1}. In what follows we can express the DBV Curve by

$$
\begin{aligned}
G := Y \circ X^{-1} : [0,1] &\to [0,1] \\
n &\to G(n) = Y(X^{-1}(n)).
\end{aligned}
\tag{9}
$$

$G(n)$ can be interpreted as the fraction of total value contributed by the top ranked nth fraction of items when the items are arranged in decreasing order of their annual usage values. Γ is given by $\{(n, G(n)) : n \in [0,1]\}$. Aggarwal proved the following properties of the DBV Curve [Agg84b].

Theorem 4.2 *The following statements hold:*

1. $G(0) = 0, \quad G(1) = 1,$
2. $G'(n) = \dfrac{r}{\mu_R} > 0$ *for all* $r \neq 0$ *with* $r = X^{-1}(n)$, $n \in [0,1]$,
3. $G''(n) = -\dfrac{1}{\mu_R f(r)} < 0$ *with* $r = X^{-1}(n)$, $n \in [0,1]$,
4. $G(n) \geq n$.

Proof.

1. $G(0) = Y(X^{-1}(0)) = Y(r_0) = 0$, $G(1) = Y(X^{-1}(1)) = Y(r_m) = 1$.

2. The composition $G = Y \circ X^{-1}$ is differentiable if Y and X^{-1} are differentiable. The differentiability of X^{-1} at $n := X(r)$, $r \in [r_m, r_0]$ results from the differentiability as well as the strong monotonicity of X. The derivative of X^{-1} in $n = X(r)$ is given by $(X^{-1})'(n) = \frac{1}{X'(r)} = \frac{1}{X'(X^{-1}(n))}$. We obtain $G'(n) = (Y \circ X^{-1})'(n) = Y'(X^{-1}(n)) \cdot (X^{-1})'(n) = Y'(r)\frac{1}{X'(r)} = -\frac{rf(r)}{\mu_R}\frac{-1}{f(r)} = \frac{r}{\mu_R} > 0$ for all $r \neq 0$.

3. $G''(n) = \dfrac{(X^{-1})'(n)}{\mu_R} = \dfrac{1}{\mu_R X'(r)} = -\dfrac{1}{\mu_R f(r)} < 0.$

4. From 2. and 3. it follows that G is concave and strictly monotonically increasing in n. Involving 1. we conclude that the diagonal $D_H := \{(x,x) : x \in [0,1]\}$ is always situated below G. □

The DBV Curve Γ is **symmetrical** with respect to the line $D_O := \{(x, 1-x) : x \in [0,1]\}$ if and only if $(n, G) \in \Gamma$ implies $(1-G, 1-n) \in \Gamma$. The point $(n^*, G^*) := (1-G, 1-n)$ is called the **symmetrical point** of (n, G) on the DBV Curve Γ. The point of intersection with the line D_O is called **point of symmetry** . For symmetrical DBV Curves there exists a special relation between the annual usage values that determine symmetrical points.

Lemma 4.3 *Symmetrical DBV Curves have the property*

$$r^* = \frac{\mu_R^2}{r}, \tag{10}$$

where r, r^* *are the annual usage values that determine two symmetrical points* $(n, G), (n^*, G^*) \in \Gamma$.

Proof. We denote the straight line through $(n, G) \in \Gamma$ and $(n^*, G^*) \in \Gamma$ by L. L is parallel to the diagonal D_H and has slope 1. Denote the tangents of the symmetrical points with Tg and Tg*. Their intersection is situated on D_O and together with L they constitute an isosceles triangle. The angle between L and Tg or Tg* is denoted by α. This implies that the slope of Tg is $\tan(45+\alpha)$ and the slope of Tg* is $\tan(45-\alpha)$. Using $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$ we conclude that

$$\tan(45+\alpha)\tan(45-\alpha) = \left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)\left(\frac{1-\tan\alpha}{1+\tan\alpha}\right) = 1.$$

Involving the second property of the DBV Curve from Theorem 4.2 proves Lemma 4.3, because $\frac{r}{\mu_R}\frac{r^*}{\mu_R} = 1$ implies $r^* = \frac{\mu_R^2}{r}$. □

Corollary 4.4 *The point of symmetry of a symmetrical DBV Curve is determined by $r_f := \mu_R$. The slope of the DBV Curve at this point is 1.*

4.2 The Objective Function Reformulated

The continuous version of the optimisation problem is as follows: for a given inventory distribution an optimal partition $\mathcal{S}^* = (S_1^*, ..., S_m^*)$ of the set $T = [r_m, r_0] \subset \overline{R}_+$ and the optimal order frequencies $F_1^*, ..., F_m^*$ of all classes have to be determined such that the expected total cost per time unit is minimal.

Keeping all the assumptions made in section 2.1, the optimal order frequency of an item with an annual usage value of r and a given cost ratio K is $F^* = \sqrt{\frac{r}{K}}$ (see (2)). There still exists an optimal grouping of the items such that all sets S_j are consecutive. Recall that for given boundaries r_j, r_{j-1} the optimal order frequency of class S_j is $F_j = \sqrt{\frac{\bar{r}(r_j, r_{j-1})}{K}}$, where $\bar{r}(r_j, r_{j-1}) := \int_{r_j}^{r_{j-1}} r f(r)\, dr \cdot (\int_{r_j}^{r_{j-1}} f(r)\, dr)^{-1}$ denotes the mean annual usage value of the items in class S_j (see (5)). The optimal boundary r_j between S_j and S_{j+1} for given order frequencies F_j, F_{j+1} is $r_j = KF_jF_{j+1}$ (see Theorem 3.1). Therefore an optimal solution is completely determined by the boundaries $r_1, ..., r_{m-1}$ or the order frequencies $F_1, ..., F_m$.

The objective function depending on the order frequencies will be denoted by $\mathrm{Z}_{\mathrm{F},c}$ (c for continuous). With $F_{m+1} := \frac{r_m}{KF_m}$ and $F_0 := \frac{r_0}{KF_1}$ the optimisation problem is given by (see (7))

$$\mathrm{Z}_{\mathrm{F},c}(F_1, ..., F_m) = \sum_{j=1}^{m} \int_{KF_{j+1}F_j}^{KF_jF_{j-1}} \left(KF_j + \frac{r}{F_j} \right) f(r)\, dr \rightarrow \min_{(F_1,...,F_m) \in R_{+,\geq}^m} .$$

If the objective function depends on the class boundaries the optimisation problem is given by (see (8))

$$\mathrm{Z}_{\mathrm{r},c}(r_1, ..., r_{m-1}) = \sum_{j=1}^{m} \sqrt{K\mu_R \int_{r_j}^{r_{j-1}} f(r)\, dr \int_{r_j}^{r_{j-1}} \frac{r}{\mu_R} f(r)\, dr} \rightarrow \min_{(r_1,...,r_{m-1}) \in T_{\geq}^{m-1}} .$$

Since T is compact the existence of a minimum follows from the fact that $\mathrm{Z}_{\mathrm{F,c}}$ and $\mathrm{Z}_{\mathrm{r,c}}$ are continuous.

The objective functions can be standardised by devision by $\sqrt{K\mu_R}$. This leads to

$$\hat{\mathrm{Z}}_{\mathrm{r},c}(r_1, ..., r_{m-1}) = \sum_{j=1}^{m} \sqrt{\int_{r_j}^{r_{j-1}} f(r)\, dr \int_{r_j}^{r_{j-1}} \frac{r}{\mu_R} f(r)\, dr} \tag{11}$$

and

$$\hat{\mathrm{Z}}_{\mathrm{F},c}(F_1, ..., F_m) = \sum_{j=1}^{m} \int_{KF_{j+1}F_j}^{KF_jF_{j-1}} \left(\sqrt{\frac{K}{\mu_R}} F_j + \frac{r}{\sqrt{K\mu_R} F_j} \right) f(r)\, dr.$$

With standardised order frequencies $\hat{F}_j := \sqrt{\frac{K}{\mu_R}} F_j$ for all $j = 1, ..., m$ this equals

$$\hat{\mathrm{Z}}_{\hat{\mathrm{F}},c}(\hat{F}_1, ..., \hat{F}_m) = \sum_{j=1}^{m} \int_{\mu_R\hat{F}_{j+1}\hat{F}_j}^{\mu_R\hat{F}_j\hat{F}_{j-1}} \left(\hat{F}_j + \frac{r}{\mu_R \hat{F}_j} \right) f(r)\, dr. \tag{12}$$

In case each item is replenished with its optimal order frequency the standardised objective function will be denoted by $\widehat{\mathrm{ZO}}_{\mathrm{c}}$. The optimal standardised order frequency of an item with an annual usage value of r is $\hat{F}^* = \sqrt{\frac{K}{\mu_R}} F^* = \sqrt{\frac{r}{\mu_R}}$ and we obtain

$$\widehat{\mathrm{ZO}}_{\mathrm{c}} = \int_{r_m}^{r_0} \sqrt{\frac{r}{\mu_R}} f(r)\, dr.$$

$\widehat{\mathrm{ZO}}_{\mathrm{c}}$ is a lower bound of $\hat{\mathrm{Z}}_{\mathrm{r},c}$ which is reached if $m \to \infty$. The upper bound of $\hat{\mathrm{Z}}_{\mathrm{r},c}$ is 1. It is reached at $m = 1$. The optimal standardised order frequency for this case is 1.

4.3 Results for symmetrical DBV Curves

It will be shown that the optimal solution of the optimisation problem for symmetrical DBV Curves is composed of complementary symmetrical class boundaries. Crouch/Oglesby [CO78] used this property implicitly for determining the optimal solution for lognormally distributed inventories. We show that the property of their solution is shared by every symmetrical DBV Curve.
Let r be the annual usage value that belongs to $(n, G) \in \Gamma$. This means that $r = X^{-1}(n)$ implies $G := G(n) = Y(X^{-1}(n))$. The symmetrical point of (n, G) is denoted by $(n^*, G^*) \in \Gamma$ or r^*. It is given by $(n^*, G^*) = (1 - G, 1 - n)$, $r^* = \frac{\mu_R^2}{r}$ (see Lemma 4.3).
From Corollary 3.2 it follows that there exists a relationship for the optimal boundary r_j between S_j and S_{j+1}:

$$r_j^2 = \bar{r}(r_j, r_{j-1}) \cdot \bar{r}(r_{j+1}, r_j). \tag{13}$$

Theorem 4.5 *Let Γ be a symmetrical DBV Curve. If the boundaries r_{j+1}, r_j, r_{j-1} satisfy (13) this also holds for r_{j+1}^*, r_j^* and r_{j-1}^*.*

Proof. Consider (n_{j-1}, G_{j-1}), $(n_j, G_j), (n_{j+1}, G_{j+1}) \in \Gamma$ and r_{j-1}, r_j, r_{j+1}, where $r_k = X^{-1}(n_k)$ implies $G_k = Y(X^{-1}(n_k))$ for all $k = j-1, j, j+1$. Define $pn_k := n_k - n_{k-1} = \int_{r_k}^{r_{k-1}} f(r)\, dr$, $pG_k := G_k - G_{k-1} = \int_{r_k}^{r_{k-1}} \frac{r}{\mu_R} f(r)\, dr$ for all $k = j, j+1$ as well as the symmetrical fractions given by $pn_k^* := n_{k-1}^* - n_k^* = \int_{r_{k-1}^*}^{r_k^*} f(r)\, dr$, $pG_k^* := G_{k-1}^* - G_k^* = \int_{r_{k-1}^*}^{r_k^*} \frac{r}{\mu_R} f(r)\, dr$ for all $k = j, j+1$. Now, obviously

$$r_j^2 = \bar{r}(r_j, r_{j-1}) \cdot \bar{r}(r_{j+1}, r_j) = \frac{\int_{r_j}^{r_{j-1}} r f(r)\, dr}{\int_{r_j}^{r_{j-1}} f(r)\, dr} \cdot \frac{\int_{r_{j+1}}^{r_j} r f(r)\, dr}{\int_{r_{j+1}}^{r_j} f(r)\, dr} = \mu_R \frac{pG_j}{pn_j} \mu_R \frac{pG_{j+1}}{pn_{j+1}}.$$

Let us now consider pn_j^* and pG_j^*. We have

$$pn_j^* := n_{j-1}^* - n_j^* = 1 - G_{j-1} - (1 - G_j) = G_j - G_{j-1} = pG_j \tag{14}$$

and

$$pG_j^* := G_{j-1}^* - G_j^* = 1 - n_{j-1} - (1 - n_j) = n_j - n_{j-1} = pn_j. \tag{15}$$

This yields

$$r_j^2 = \mu_R \frac{pn_j^*}{pG_j^*} \mu_R \frac{pn_{j+1}^*}{pG_{j+1}^*},$$

and equivalently

$$\frac{\mu_R^4}{r_j^2} = \mu_R \frac{pG_j^*}{pn_j^*} \mu_R \frac{pG_{j+1}^*}{pn_{j+1}^*}.$$

The relation $r_j^* = \frac{\mu_R^2}{r_j}$ from Lemma 4.3 proves the theorem. □

Definition 4.6 *Let Γ be a symmetrical DBV Curve. A class is called the* ***symmetrical class*** *of S_j if and only if it has boundaries $r_{j-1}^* = \frac{\mu_R^2}{r_{j-1}}$ and $r_j^* = \frac{\mu_R^2}{r_j}$, where r_j and r_{j-1} are the boundaries of S_j. The order frequency of the symmetrical class of S_j is denoted by F_{j^*}, the objective function by $\mathrm{Z}_{\mathrm{r,c,j^*}}$.*

Because of $pn_j^* = pG_j$ and $pG_j^* = pn_j$ (see (14) and (15)) the following holds:

Corollary 4.7

$$\begin{aligned} \hat{\mathrm{Z}}_{\mathrm{r},c,j}(r_j, r_{j-1}) &= \hat{\mathrm{Z}}_{\mathrm{r},c,j^*}(r_{j-1}^*, r_j^*), \\ \hat{F}_j &= \frac{1}{\hat{F}_{j^*}}. \end{aligned}$$

We have proved that the value of the objective function is the same for symmetrical classes. From this it follows that the optimal solution for $[r_m, r_f]$ implies optimality of the solution given by symmetrical classes for $[r_f, r_0]$ and vice versa ($r_f = \mu_R$). The most important result of this section is that searching for the solution of the optimisation problem can be restricted to $[r_m, r_f]$ or $[r_f, r_0]$ for symmetrical DBV Curves. This reduces the number of decision variables by the factor of 2. The optimal boundaries of a symmetrical DBV Curve satisfy

$$r_{m-j} = \frac{\mu_R^2}{r_j}, \; j = 1, ..., m-1.$$

It follows that

$$\begin{aligned} r_{\frac{m}{2}} &= \mu_R \quad \text{for } m \text{ even}, \\ F_{\frac{m+1}{2}} &= \sqrt{\tfrac{\mu_R}{K}} \quad \text{for } m \text{ odd}. \end{aligned}$$

4.4 DBV Curves

Throughout many industries in all countries, the DBV Curves of the inventory follow very similar patterns, which resemble curves generated by lognormal distributions [Bro77], [Agg84b]. The lognormal distribution is commonly suggested for modeling [Her76] but intricate to manipulate. Therefore Aggarwal gave an alternative proposal: the Pareto distribution [Agg84b]. Three different forms for the Pareto DBV Curve have been suggested: skewed Pareto, symmetric Pareto and Power Pareto. The symmetric and the Power Pareto Curves are well suited for indepth investigation and represent a wide class of inventories.

4.4.1 The Lognormal DBV Curve

Assume that the probability density function f of the inventory distribution is lognormal with parameters μ and σ^2. Then

$$R \sim \Lambda_{\mu,\sigma^2}, \quad f(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}\exp(-\frac{(\ln x - \mu)^2}{2\sigma^2}) \text{ for all } x > 0,$$

$$\mu_R = \exp(\mu + \frac{\sigma^2}{2}), \quad \sigma_R = \exp(\mu + \frac{\sigma^2}{2})\sqrt{\exp\sigma^2 - 1}$$

and

$$
\begin{aligned}
X(r) &= \int_r^\infty \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}\exp(-\frac{(\ln x-\mu)^2}{2\sigma^2})\,dx \\
&= \int_{\frac{\ln r-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})\,dy,
\end{aligned}
$$

$$
\begin{aligned}
Y(r) &= \frac{1}{\exp(\mu+\frac{\sigma^2}{2})}\int_r^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(\ln x-\mu)^2}{2\sigma^2})\,dx \\
&= \int_r^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(\ln x-\mu)^2}{2\sigma^2}-\mu-\frac{\sigma^2}{2})\,dx \\
&= \int_{\frac{\ln r-\mu-\sigma^2}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})\,dy,
\end{aligned}
$$

using the substitution $y=\frac{\ln x-\mu-\sigma^2}{\sigma}$, $dy=\frac{1}{\sigma x}dx$ in the last step. This yields

$$X(r)=1-\Phi(\frac{\ln r-\mu}{\sigma}), \qquad Y(r)=1-\Phi(\frac{\ln r-\mu-\sigma^2}{\sigma}).$$

Elimination of r leads to

$$G(n)=1-\Phi(\Phi^{-1}(1-n)-\sigma). \tag{16}$$

The Lognormal DBV Curve only depends on σ. σ can be interpreted as a measure of variability. The Lognormal DBV Curve is symmetrical with respect to the line D_O. Its point of symmetry is given by

$$n_f=1-\Phi(\frac{\sigma}{2}), \quad G_f=\Phi(\frac{\sigma}{2})$$

(see [Agg84b]).

4.4.2 Pareto DBV Curves

4.3.2.1 The skewed Pareto DBV Curve

The skewed Pareto DBV Curve is defined by the density function

$$f(x)=\begin{cases} vr_m^v x^{-(1+v)} & x\ge r_m \\ 0 & x<r_m \end{cases}$$

where $v>0$ is the Pareto parameter and $r_m>0$ is a scale parameter. For $v>1$ the mean μ_R is defined and

$$
\begin{aligned}
X(r) &= \int_r^\infty vr_m^v x^{-(1+v)}\,dx &&= (\frac{r_m}{r})^v &&\text{if } r\ge r_m, \\
Y(r) &= \frac{1}{\mu_R}\int_r^\infty vr_m^v x^{-v}\,dx &&= (\frac{r_m}{r})^{v-1} &&\text{if } r\ge r_m.
\end{aligned}
$$

Eliminating r yields $G(n)=n^\lambda$ with $\lambda=\frac{v-1}{v}$. The skewed Pareto DBV Curve is not symmetrical.

4.3.2.2 The symmetric Pareto DBV Curve

The equation of the symmetric Pareto DBV Curve is given by $G(n) = \frac{(1+\theta)^2 n}{(1-\theta)^2+4\theta n}, 0 \leq n \leq 1$ with variability measure $0 < \theta < 1$ [Agg84b]. It has all the properties of a DBV Curve. Using 2. from Theorem 4.2 it can be shown that the underlying density function is given by

$$f(r) = -X'(r) = \frac{1-\theta^2}{8\theta}\sqrt{\mu_R} \cdot r^{-\frac{3}{2}},\, r \in [r_m, r_0]. \tag{17}$$

The Pareto DBV Curve is symmetrical with respect to the line D_O. Its point of symmetry is given by $(n_f, G_f) := (\frac{1-\theta}{2}, \frac{1+\theta}{2})$.

4.3.2.3 The Power Pareto Curve

The equation for the Power Pareto Curve has three parameters. This provides high flexibility to guarantee good approximation for the entire range of the observed inventory distribution [Agg84b]. The functional form of the DBV Curve is expressed in the $\pi - \eta-$coordinate system, where for any $(n, G) \in [0, 1]^2$

$$\pi = \frac{G+n}{\sqrt{2}},\, \eta = \frac{G-n}{\sqrt{2}}. \tag{18}$$

The equation for the DBV Curve for $0 \leq \pi \leq \sqrt{2}$ is assumed to be

$$\begin{aligned} \eta &= g(\pi) \\ \text{with} \quad g(\pi) &= a\pi^\alpha(\sqrt{2}-\pi)^\beta,\ a > 0,\, 0 < \alpha, \beta \leq 1. \end{aligned} \tag{19}$$

The Power DBV Curve is symmetrical if and only if $g(\pi) = g(\sqrt{2} - \pi)$ for all $\pi < \frac{\sqrt{2}}{2}$. This is equivalent to $\alpha = \beta$.

4.5 Optimal Solution

4.5.1 The symmetric Pareto DBV Curve

Aggarwal [Agg83] has shown that the optimal class boundaries can be obtained in closed-form fashion if the inventory distributions are modeled by a symmetric Pareto DBV Curve. He used a different representation of the symmetric Pareto DBV Curve but the proof is the same for $G(n) = \frac{(1+\theta)^2 n}{(1-\theta)^2+4\theta n}$, $0 \leq n \leq 1$ where $0 < \theta < 1$. Several properties of the solution will be quoted. We work with $\hat{\mathrm{Z}}_\mathrm{r}$ and denote it by $\hat{\mathrm{Z}}$.

Theorem 4.8 *Given the number of classes m, the optimal solution of the optimisation problem has the property*

$$\hat{\mathrm{Z}}^*_\mathrm{j} = \frac{\hat{\mathrm{Z}}^*}{\mathrm{m}} \ \textit{for all } \mathrm{j} = 1, ..., \mathrm{m}.$$

Corollary 4.9 *The class boundaries are complementary symmetrical points on the Pareto DBV Curve, given by* $n^*_j = P[S^{\frac{j}{m}} - 1]$ *and* $G^*_j = R[1 - S^{-\frac{j}{m}}]$ *for all* $j = 1, ..., m-1$, *where* $P := \dfrac{(1-\theta)^2}{4\theta}$, $R := \dfrac{(1+\theta)^2}{4\theta}$ *and* $S := \dfrac{(1+\theta)^2}{(1-\theta)^2}$.

Theorem 4.10 (Properties of the solution) *The optimal values of pn_j (j=1, ..., m), pG_j, $\hat{F}_j$ and r_j over the successive classes form geometric sequences*

$$\begin{aligned} pn_j &= n_1 S^{\frac{j-1}{m}}, \\ pG_j &= G_1 S^{\frac{1-j}{m}}, \\ \hat{F}_j &= S^{\frac{m+1-2j}{2m}}, \\ r_j &= \mu_R S^{\frac{m-2j}{m}}, \\ \bar{r}(r_j, r_{j-1}) &= \mu_R S^{\frac{m+1-2j}{m}}. \end{aligned}$$

Based on practical experiences the KCM uses a geometric progression of the order frequencies because it turned out to perform better than e.g. a linear progression [RBD94].
For given θ the optimal boundaries can be easily calculated. Estimating θ requires an effort of $O(n)$. The computational effort for establishing the grouping can therefore be reduced considerably compared to the algorithms found in [Cha81], [COR82], [Agg84a], [Cha85] and [Bas86].

4.5.2 The Lognormal DBV Curve

Closed-form results have not been obtained if the inventory distribution is lognormal. However, by making use of the symmetry of the lognormal DBV Curve the effort for determining a solution can be reduced considerably (see section 4.3). The objective function that has to be minimized is

$$\mathrm{Z_F}(F_1, ..., F_m) = \sum_{j=1}^{m} \int_{KF_j F_{j+1}}^{KF_{j-1} F_j} (KF_j + \frac{x}{F_j}) \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} \exp(-\frac{(\ln x - \mu)^2}{2\sigma^2})\, dx, \tag{20}$$

where $F_{m+1} = 0$ and $F_0 = \infty$ or

$$\mathrm{Z_r}(r_1, ..., r_{m-1}) = \sum_{j=1}^{m} \sqrt{\frac{K}{2\pi\sigma^2} \int_{r_j}^{r_{j-1}} \frac{1}{x} \exp(-\frac{(\ln x - \mu)^2}{2\sigma^2})\, dx \int_{r_j}^{r_{j-1}} \exp(-\frac{(\ln x - \mu)^2}{2\sigma^2})\, dx} \tag{21}$$

with $r_m = 0$ and $r_0 = \infty$.

Starting with initial values for all elements of r = $(r_1, ..., r_{m-1})$ the value of the objective function (21) can be reduced iteratively by using a direct search method [Him72]. The simplest way is to change one variable at a time while keeping the others constant until a minimum is reached. By doing this the value of the objective function is decreasing gradually and a local minimum is found. This follows from $\mathrm{Z_r}$ being bounded from below by $\sqrt{K\mu_R}\widehat{\mathrm{ZO}} = \sqrt{K\mu_R}\exp(-\frac{\sigma^2}{8})$. Using different starting points a high reliability of the solution can be achieved.
Crouch/Oglesby tabulated the results for a lognormal distribution [CO78]. If the items are classified into five or more classes the solution for a lognormal distribution has none of the properties of the solution for the symmetric Pareto DBV Curve shown in Theorem 4.10.

5 Comparison and Concluding Remarks

A comparison of the optimal solution for the optimisation problem under consideration and the solution obtained by using the K Curve Methodology to set class boundaries reveals a fundamental difference in how the items are allocated within classes. The optimal solution follows the philosophy

of the ABC-Analysis: the items are sorted in decreasing order of their annual usage values and fixed percentages of the items fall into the m classes. This classification is independent of the costs C and I. The optimal order frequencies of the classes however depend on the cost ratio $K = \frac{2C}{I}$ (see (5)).
The KCM proceeds differently. The order frequencies are chosen in form of a geometric series with factor 2 and the appropriate value of K for the range of items in consideration is determined by evaluating the K-Curve. The class boundaries are set optimally with respect to the order frequencies of the m classes. If management decides that the appropriate value of K has changed the class boundaries will change according to K. In consequence also the percentage of items that fall into the different classes is changing.

However, the KCM has additional advantages which are not represented by the modeling features used in the optimization problem. Nevertheless these features should be taken into consideration if a multi-item inventory management system is implemented.
First, the KCM is a very simple technique to set class boundaries and allocate items within classes. The order frequencies of the classes can be chosen freely such that a practical order frequency progression like weekly, fortnightly, four weekly, eight weekly and so on can be implemented.
Second, it is difficult if not impossible to attribute true values to stockholding and ordering costs in practice. Therefore, the optimal solution when using such costs is a theoretical optimum. By using the KCM some of the difficulties in prescribing these costs for each individual item are avoided because the total inventory performance of a group of items is considered. The objectives of management can be defined in terms of the inventory level and workload. In a second step the technical parameters used to reach these objectives can be set by determining the appropriate value of K.
Third, the parameters set might not be constant over a long time period. In contrast to the classical ABC inventory classification and the optimal solution the KCM is able to respond to a dynamic environment characterised by shorter product life-cycles and market driven demands. This is due to the fact that the KCM classifies items individually by considering their annual usage values whereas the optimal solution classifies a fixed percentage of the items into the different classes. As a result a new optimization problem has to be solved each time the parameters have changed.
The objective function derived from the EOQ model that has been used in the theoretical papers (see [Cha81], [Agg83], [Bas86]) does not respond to these requirements. First, it is difficult to determine the order frequencies because of the difficulties in establishing the value of K. Second, it may be impossible to implement the optimal order frequencies in real-life because the objective function does not take into consideration that not all order frequencies are realizable in practice. Third, given the fact that product life-cycles get shorter and shorter an update of the classification has to be made in short intervals. Each time a change of the optimal order frequencies has to be carried out which is considered expensive by practitioners.

Applying the KCM it is however possible that all the items fall either into the first or the last class depending on a very low or a very high value of K. Obviously this classification is far away from an optimal grouping of the items. This shows that keeping the same order frequencies for all values of K can lead to a poor ordering policy. Therefore a regular control of the KCM should ensure that for changing conditions the total inventory costs do not deviate excessively from the optimum. If this is the case the order frequencies should be adjusted.
In section 3 it has been shown that the KCM does not come to an optimal grouping of the items because the order frequencies are not subject of an optimisation. Fortunately, one of the main strenghts of the EOQ analysis is that costs rise slowly for order frequencies near to optimal (see

[Ter94] p. 96). Having determined the optimal order frequencies it is thus possible to smooth them for obtaining practical values without going away from the optimal costs very far.

The insensitivity of the inventory cost minimization problem in consideration, which stems from using the EOQ model and which may be attributed to the KCM as well is demonstrated by the following example: We have a symmetric Pareto DBV Curve of the products under consideration and the products have to be classified into $m = 6$ classes. The mean annual usage value of the products can be normalized $\mu_R = 1$. As variability measure θ we choose three different values $\theta = 0.5, 0.8, 0.9$ which are realistic for the retail industry, industrial producers and the hightechnology industry respectively (see [Agg84c]). The optimal order frequencies of the classes are given by $F_j = \frac{1}{\sqrt{K}} (\frac{1+\theta}{1-\theta})^{\frac{m*1-2j}{m}}, j = 1, ..., m$ (see Theorem 4.10). Table 1 shows the values of $\frac{1}{F_j}, j = 1, ..., m$ for $\theta = 0.5, 0.8$ and 0.9 if we assume the costs to be such that $K = 250$.

$\theta =$	0.5	0.8	0.9
F_1^{-1}	6.329	2.534	1.359
F_2^{-1}	9.129	5.270	3.627
F_3^{-1}	13.166	10.963	9.679
F_4^{-1}	18.989	22.804	25.828
F_5^{-1}	27.386	47.434	68.920
F_6^{-1}	39.498	98.667	183.907

Table 1: Optimal order periods

Ordering the items optimally, i.e. the items of class 1 have to be replenished every 6.329 days if $\theta = 0.5$, can not be realized in practice. Therefore these numbers are rounded.

$\theta =$	0.5	0.8	0.9
F_1^{-1}	5	2.5	2
F_2^{-1}	10	5	5
F_3^{-1}	15	10	10
F_4^{-1}	20	20	25
F_5^{-1}	25	50	70
F_6^{-1}	40	100	200

Table 2: Rounded order periods

We can now choose these order frequencies to be used when applying the K-Curve Methodology to the different inventories. Working with these order frequencies might be inefficient when the value of K changes. We are therefore interested in how important the difference between the optimal value of the cost function Z_o and the realized costs Z_r for different K-values is. Table 3 gives upper and lower bounds for K if a certain relative deviation from the optimal costs must not be exceeded.

As table 3 shows the intervals for K are surprisingly wide. Within these intervals for K the K-Curve Methodology represents a practical and only slightly suboptimal inventory management

max. rel. deviation between Z_o and Z_r	upper and lower bounds of K		
	$\theta = 0.5$	0.8	0.9
1%	479.4	488.8	635.0
	90.7	127.9	251.5
2%	655.3	657.4	848.1
	66.6	95.1	188.5
5%	1014.2	1146.9	1410.8
	41.3	54.5	113.4
10%	1563.5	1925.0	2398.0
	26.4	32.5	66.7

Table 3: Bounds for K

policy. If another K has to be applied it would be beneficial to change the order frequencies of the classes.

We conclude that although the KCM does not represent the optimal inventory policy for the objective function that usually has been treated, our investigation reveals why it nevertheless may be considered to work well. Furthermore this objective function does not take into account that an adjustment of the order frequencies to changing costs or product ranges is considered to be impractical and expensive by practicioners. Therefore there is good reason to use an improved KCM where a regular control of the ordering policy is ensured.

References

[Agg83] AGGARWAL, V. (1983): *A Closed-Form Approach for Multi-Item Inventory Grouping.* Naval Research Logistics Quarterly 30, 471-485.

[Agg84a] AGGARWAL, V. (1984): *Grouping Multi-Item Inventory using Common Cycle Periods.* European Journal of Operational Research 17, 369-372.

[Agg84b] AGGARWAL, V. (1984): *Robustness of the Partitions in Grouping Multi-Item Inventories.* International Journal of Productions Research 22, 923-935.

[Agg84c] AGGARWAL, V. (1984): *Modeling of Distributions by Value for Multi-Item Inventory.* IIE Transactions 16, 90-98.

[AE88] AKSOY, Y., S. ERENGUC (1988): *Multi-Item Inventory Models with Coordinated Replenishments: A Survey.* Inventory Journal of Production Management 8, 63-73.

[Bas86] BASTIAN, M. (1986): *Joint Replenishment in Multi-Item Inventory Systems.* Journal of the Operational Research Society 37, 1113-1120.

[Bro77] BROWN, R.G. (1977): *Materials Management-Systems.* Wiley, New York.

[BDR95] BURCHER, P., S. DUPERNEX, G. RELPH (1995): *Competitive Adnatage for Batch Manufaturing using Capacitated Batch Sizing within MRP.* BPICS Control, 25-30.

[Cha81] CHAKRAVARTY, A.K. (1981): *Multi-Item Inventory Aggregation into Groups.* Journal of the Operational Research Society 32, 19-26.

[Cha85] CHAKRAVARTY, A.K. (1985): *An Optimal Heuristic for Coordinated Multi-Item Inventory Replenishments.* Journal of the Operational Research Society 36, 1027-1039.

[COR82] CHAKRAVARTY, A.K., J.B.ORLIN, U.G.ROTHBLUM (1982): *A Partitioning Problem with Additive Objective with Application to Optimal Inventory Groupings for Joint Replenishment.* Operations Research 30, 1018-1022.

[CO78] CROUCH, R.B., S. OGLESBY (1978): *Optimization of a Few Lot Sizes to Cover a Range of Requirements.* Journal of the Operational Research Society 29, 897-904.

[Don81] DONALDSON, W.A. (1981): *Grouping of Inventory Items by Review Period.* Journal of the Operational Research Society 32, 1075-1076.

[GS89] GOYAL, S.K., A.T. SATIR (1989): *Joint Replenishment Inventory Control: Deterministic and Stochastic Models.* European Journal of Operational Research 38, 2-13.

[Him72] HIMMELBLAU, D.M. (1972): *Applied Nonlinear Programming.* Mc Graw Hill, New York.

[Her76] HERRON, D. (1976): *Industrial Engineering Applications of ABC Curves.* AIIE Transactions 8, 210-218.

[LR95] LENARD, J.D., B. ROY (1995): *Multi-Item Inventory Control: A Multicriteria View.* European Journal of Operational Research 87, 685-692.

[PP76] PAGE, E., R.J. PAUL (1976): *Multi-Product Inventory Situations with One Restriction.* Operational Research Quaterly 27, 815-834.

[RBD94] RELPH, G., L.BROWNE, S. DUPERNEX, R. DANCE (1994): *Improved Inventory Performance into the 21st Century.* BPICS Control, 23-26.

[SBR90] SHAH, S., P. BUCHER, G. RELPH (1990): *Extending the Pareto Principle to MRP Controlled Parts and Regaining MRP Control.* BPICS Control, 39-45.

[Ter94] TERSINE, R. (1994): *Principles of Inventory and Materials Management.* 4th edition, Prentice-Hall, Englewood Cliffs.